Light-cone gauge string field theory in noncritical dimensions

JHEP12(2009)010

(http://iopscience.iop.org/1126-6708/2009/12/010)

Published by IOP Publishing for SISSA

Received: *October 9, 2009*
Accepted: *November 13, 2009*
Published: *December 2, 2009*

Light-cone gauge string field theory in noncritical dimensions

Yutaka Baba,[a] Nobuyuki Ishibashi[b] and Koichi Murakami[a]

[a] *Theoretical Physics Laboratory, RIKEN, Wako, Saitama 351-0198, Japan*
[b] *Institute of Physics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan*

E-mail: ybaba@riken.jp, ishibash@het.ph.tsukuba.ac.jp, murakami@riken.jp

Abstract: We study light-cone gauge string field theory in noncritical space-time dimensions. Such a theory corresponds to a string theory in a Lorentz noninvariant background. We identify the worldsheet theory for the longitudinal coordinate variables $X^{\pm}$ and study its properties. It is a CFT with the right value of Virasoro central charge, using which we propose a BRST invariant formulation of the worldsheet theory.

Keywords: Conformal Field Models in String Theory, String Field Theory, BRST Symmetry

ArXiv ePrint: 0909.4675

 doi:10.1088/1126-6708/2009/12/010

Contents

1 Introduction

Light-cone gauge string field theory provides a useful way to define string theories [1–3]. Given the action, it is possible to define the amplitudes and calculate them perturbatively, although we should check if they are well-defined. Since it is a gauge fixed theory, there is no problem in considering a light-cone gauge string field theory in noncritical space-time dimensions. Being noncritical, it does not possess the space-time Lorentz invariance. It should correspond to a string theory in a Lorentz noninvariant background. In other words, we should be able to find a BRST invariant worldsheet theory, with a nonstandard $X^{\pm}$ part.

What we would like to do in this paper is to study this $X^{\pm}$ theory. We give the energy-momentum tensor and the action of the theory and calculate the correlation functions. We show that the energy-momentum tensor possesses desired properties, and construct a BRST invariant formulation of the worldsheet theory.

The reason why we are interested in this theory is that it can be used to regularize the string field theory. The dimensional regularization is one of the most powerful regularizations for ordinary quantum field theory and it may be useful also in string theory.

In particular, in the light-cone gauge superstring field theory, unwanted divergences occur even at the tree level, because of the transverse supercurrent insertions at the interaction points of the vertices. In ref. [4], we have proposed a dimensional regularization scheme to deal with these divergences. We have found that the divergences of the tree amplitudes can be regularized by shifting the number of the space-time dimensions. We have checked that the results of the first quantized formalism are reproduced without any counterterms for the four-point case. In order to proceed further we need to show that the dimensional regularization preserves important symmetries of the theory. If the light-cone gauge string field theory corresponds to a BRST invariant formulation even in noncritical dimensions, it means that the dimensional regularization preserves the BRST symmetry.

In this paper, we deal with the case of closed bosonic string field theory. We restrict ourselves to the tree amplitude and consider the worldsheet theory on the complex plane. The organization of this paper is as follows. In section 2, we present a way to rewrite light-cone gauge string amplitudes in a BRST invariant form. We follow the procedure in noncritical dimensions, and show what kind of worldsheet theory of $X^{\pm}$ should appear in the end. In section 3, we study the theory for $X^{\pm}$ and show that it is a CFT with the right Virasoro central charge. Namely, combining the CFT for $X^{\pm}$ constructed here with the worldsheet theory for the light-cone gauge strings in noncritical dimensions, we obtain a CFT with the central charge 26 by which we can define a BRST invariant worldsheet theory. In section 4, we show that the tree amplitudes in the noncritical case can be written in a BRST invariant form. Section 5 is devoted to discussions. In appendix A, we present the action of the light-cone gauge bosonic string field theory in d dimensions. In appendix B, the Mandelstam mapping is given. In appendix C, we present a way to calculate $\Gamma[\phi]$, which is used in sections 2 and 3. In appendix D, the details of the calculations in section 3 are presented.

2 Relation between light-cone gauge amplitudes and covariant ones

2.1 Critical case

In order to study the noncritical case, it is useful to consider the relation between light-cone gauge amplitudes and covariant ones for critical strings. One can calculate the amplitudes starting from the light-cone gauge string field theory action given in appendix A. The tree amplitudes can be expressed by path integral on the light-cone diagrams, on which the complex ρ coordinate is introduced as usual. Via the Mandelstam mapping $\rho(z)$, which is given in appendix B, one can express them using correlation functions of vertex operators on the z-plane endowed with the metric

$$ds^2 = d\rho d\bar{\rho} = e^{\phi} dz d\bar{z} \,, \qquad \phi = \ln\left(\partial\rho\bar{\partial}\bar{\rho}\right) \,. \tag{2.1}$$

The correlation functions we should consider are

$$F = (2\pi)^2 \delta\Big(\sum_{r=1}^{N} p_r^+\Big) \delta\Big(\sum_{r=1}^{N} p_r^-\Big) \int \left[dX^i\right]_{\phi} e^{-S_{X^i}} \prod_{r=1}^{N} V_r^{\mathrm{LC}} \,. \tag{2.2}$$

Here S_{X^i} denotes the free action for the 24 transverse coordinates X^i and V_r^{LC} denotes the vertex operator. We assume that the r-th external state is of the form

$$\alpha_{-n_1}^{i_1(r)} \cdots \tilde{\alpha}_{-\tilde{n}_1}^{\tilde{i}_1(r)} \cdots |p^-, p^i\rangle_r \,, \tag{2.3}$$

and the vertex operator should be

$$V_r^{\mathrm{LC}} \sim \alpha_r \frac{i\partial^{n_1} X^{i_1(r)}(w_r)}{(n_1-1)!} \cdots \frac{i\bar{\partial}^{\tilde{n}_1} X^{\tilde{i}_1(r)}(\bar{w}_r)}{(\tilde{n}_1-1)!} \cdots e^{ip_r^i X^i - p_r^- \tau_0^{(r)}}(w_r, \bar{w}_r)\Bigg|_{w_r=\bar{w}_r=0} \,, \tag{2.4}$$

where $\tau_0^{(r)}$ is defined in eq. (B.3), w_r is the coordinate of the unit disk of the r-th external string given in eq. (B.1) and $\alpha_r = 2p_r^+$. The on-shell and the level-matching conditions require that

$$\frac{1}{2}\left(-2p_r^+ p_r^- + p_r^i p_r^i\right) + \mathcal{N}_r = 1 \,, \qquad \mathcal{N}_r \equiv \sum_i n_i = \sum_j \tilde{n}_j \,. \tag{2.5}$$

The path integral measure $\left[dX^i\right]_\phi$ should be defined using the metric (2.1). It is related to the measure $\left[dX^i\right]$ which is defined with the flat metric $ds^2 = dzd\bar{z}$ as

$$\left[dX^i\right]_\phi \propto \left[dX^i\right] e^{-\Gamma[\phi]} \,,$$

Roughly speaking, $\Gamma[\phi]$ is given by the Liouville action

$$\Gamma[\phi] \sim -\frac{24}{24\pi} \int d^2z \, \partial\phi\bar{\partial}\phi \,, \tag{2.6}$$

where $d^2z = d(\mathrm{Re}\, z)d(\mathrm{Im}\, z)$. Since ϕ diverges at the poles and zeros of $\partial\rho$ by the definition (2.1), the Liouville action is not well-defined at these points. $\Gamma[\phi]$ can be evaluated by regularizing the divergences and carefully taking various effects into account [5, 6]. We present an alternative derivation of $\Gamma[\phi]$ in appendix C and give an explicit form in eq. (C.22).[1] Thus eq. (2.2) can be rewritten as

$$F \sim (2\pi)^2 \delta\left(\sum_{r=1}^N p_r^+\right) \delta\left(\sum_{r=1}^N p_r^-\right) \int \left[dX^i\right] e^{-S_{X^i} - \Gamma[\phi]} \prod_r V_r^{\mathrm{LC}} \,. \tag{2.7}$$

[1] As is mentioned in appendix C, this form of $\Gamma[\phi]$ is only up to a constant which can be fixed by factorization. With the string field action in eq. (A.1) and the $\Gamma[\phi]$ in eq. (C.22), one can see that the factor is fixed as [4]

$$\left[dX^i\right]_\phi \sim \left[dX^i\right] \mathrm{sgn}\left(\prod_{r=1}^N \alpha_r\right) e^{-\Gamma[\phi]} \,,$$

up to a numerical factor. We ignore the phase factor $\mathrm{sgn}\left(\prod_{r=1}^N \alpha_r\right)$ in the following, because it does not play any important roles.

JHEP12(2009)010

Longitudinal coordinates. In order to covariantize the amplitudes, we need to introduce the longitudinal coordinates. The light-cone gauge condition implies that X^+ equals the Lorentzian time on the light-cone diagram, which means on the z-plane,

$$X^+(z,\bar{z}) = -\frac{i}{2}\left(\rho(z)+\bar{\rho}(\bar{z})\right) \ . \tag{2.8}$$

Therefore we introduce the variable X^+ with the delta function $\delta\left(X^+ + \frac{i}{2}(\rho+\bar{\rho})\right)$, which can be expressed as

$$\begin{aligned}\delta\left(X^+ + \frac{i}{2}(\rho+\bar{\rho})\right) \sim \int \left[dX'^-\right] e^{-\frac{1}{\pi}\int d^2z X'^-\partial\bar{\partial}X^+} \prod_{r=1}^{N} e^{-ip_r^+X'^-}\left(Z_r,\bar{Z}_r\right) \\ \times \int \left[db'dc'd\tilde{b}'d\tilde{c}'\right] e^{-\frac{1}{\pi}\int d^2z\left(b'\bar{\partial}c'+\tilde{b}'\partial\tilde{c}'\right)} c'(\infty)\tilde{c}'(\infty) \ .\end{aligned} \tag{2.9}$$

Eq. (2.9) should be considered as the formal Euclideanized version of a Lorentzian path integral.[2] The Grassmann odd fields $b', c', \tilde{b}', \tilde{c}'$ of conformal weights $(1,0)$, $(0,0)$, $(0,1)$, $(0,0)$ are introduced to cancel the determinant factor $\left(\det\partial\bar{\partial}\right)^{-1}$.

With these variables, we can rewrite the right hand side of eq. (2.7) as

$$\begin{aligned}F \sim \frac{2\pi\delta\left(\sum_{r=1}^{N} p_r^-\right)}{2\pi\delta(0)} \int \left[dX^+dX'^-dX^i db'dc'd\tilde{b}'d\tilde{c}'\right] e^{-S_{X^i}-S'_\pm-S_{b'c'}} \\ \times c'(\infty)\tilde{c}'(\infty)\prod_{r=1}^{N}\left(V_r^{\mathrm{LC}} e^{-ip_r^+X'^-}\left(Z_r,\bar{Z}_r\right)\right) \ ,\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}S_{b'c'} &= \frac{1}{\pi}\int d^2z\left(b'\bar{\partial}c' + \tilde{b}'\partial\tilde{c}'\right) \ , \\ S'_\pm &= -\frac{1}{2\pi}\int d^2z\left(\partial X^+\bar{\partial}X'^- + \bar{\partial}X^+\partial X'^-\right) + \Gamma\left[\ln\left(-4\partial X^+\bar{\partial}X^+\right)\right] \ .\end{aligned} \tag{2.11}$$

We consider $S_{X^i} + S'_\pm + S_{b'c'}$ as the worldsheet action for these variables. The action $S'_\pm$ is with the interaction term which depends only on X^+. This interaction term does not affect the correlation functions with less than two insertions of X'^-, and they coincide with those of the free theory. Thus one can show the OPE

$$\begin{aligned}X^+(z,\bar{z})X^+(z',\bar{z}') &\sim \text{regular} \, , \\ X^+(z,\bar{z})X'^-(z',\bar{z}') &\sim \ln\left|z-z'\right|^2 \ .\end{aligned} \tag{2.12}$$

From the worldsheet action, the energy-momentum tensor can be obtained as

$$T(z) = \partial X^+\partial X'^- - 2\left\{X^+, z\right\} - \frac{1}{2}\partial X^i\partial X^i - b'\partial c' \ , \tag{2.13}$$

[2] In the light-cone gauge, the worldsheet should be inherently with Lorentzian signature. In this paper, we use the Euclideanized expressions, which look more familiar.

when $z \neq Z_r, z_I$. Here

$$\begin{aligned}
\{X^+, z\} &= \frac{\partial^3 X^+}{\partial X^+} - \frac{3}{2}\left(\frac{\partial^2 X^+}{\partial X^+}\right)^2 \\
&= -\frac{1}{2}\left(\partial\left(\ln\left(-4\partial X^+\bar{\partial}X^+\right)\right)\right)^2 + \partial^2\left(\ln\left(-4\partial X^+\bar{\partial}X^+\right)\right) \ ,
\end{aligned}$$

denotes the Schwarzian derivative. This term can be derived from the variation of Γ which coincides with the Liouville action except for the singular points.

Covariant variables. The covariant expression for the amplitudes can be obtained by introducing [7]

$$\begin{aligned}
b &\equiv \partial X^+ b' \ , \\
c &\equiv \left(\partial X^+\right)^{-1} c' \ ,
\end{aligned} \tag{2.14}$$

and their anti-holomorphic counterparts $\tilde{b}$ and $\tilde{c}$. The fields b and c have now weights $(2,0)$ and $(-1,0)$ respectively. We should also introduce

$$X^- \equiv X'^- + \frac{b'c'}{\partial X^+} - \frac{\partial^2 X^+}{2\left(\partial X^+\right)^2} + \frac{\tilde{b}'\tilde{c}'}{\bar{\partial}X^+} - \frac{\bar{\partial}^2 X^+}{2\left(\bar{\partial}X^+\right)^2} \ , \tag{2.15}$$

so that the OPE's between X^- and the ghosts are regular and the energy-momentum tensor (2.13) takes the form

$$T(z) = \partial X^+ \partial X^- - \frac{1}{2}\partial X^i \partial X^i - 2b\partial c - \partial b c \ , \tag{2.16}$$

which coincides with the energy-momentum tensor in the conformal gauge. In the following, we would like to rewrite the correlation function (2.10) using these new variables.

DDF operators. Let us first consider the vertex operator. From eq. (2.16), one can see that the action for the new variables should be the free action. X^- appears essentially in the same way as X'^- does in eq. (2.9), and we obtain the delta function. Therefore $\rho(z)$ and $\bar{\rho}(\bar{z})$ which appear in the integrand can be replaced by $2iX_L^+(z)$ and $2iX_R^+(\bar{z})$, where $X_L^+(z)$ and $X_R^+(\bar{z})$ are the holomorphic and the anti-holomorphic part of X^+ respectively. We will denote the equality which holds under this identification by $\approx$. Thus the factor $\left.\frac{i\partial^n X^i(w_r)}{(n-1)!}\right|_{w_r=0}$ in the definition (2.4) of V_r^{LC} can be rewritten as

$$\begin{aligned}
\left.\frac{i\partial^n X^i(w_r)}{(n-1)!}\right|_{w_r=0} &= \oint_0 \frac{dw_r}{2\pi i} i\partial X^i(w_r) w_r^{-n} \\
&\approx \oint_{Z_r} \frac{dz}{2\pi i} i\partial X^i(z) e^{-in\frac{X_L^+(z)}{p_r^+} + n\frac{\tau_0^{(r)} + i\beta_r}{\alpha_r}} \\
&= A_{-n}^{(r)i} e^{n\frac{\tau_0^{(r)} + i\beta_r}{\alpha_r}} \ ,
\end{aligned} \tag{2.17}$$

where $A_{-n}^{(r)i}$ is the DDF operator given by

$$A_{-n}^{(r)i} \equiv \oint_{Z_r} \frac{dz}{2\pi i} i\partial X^i(z) e^{-in\frac{X_L^+(z)}{p_r^+}} \ . \tag{2.18}$$

One can also use

$$e^{-i\left(p_r^- - \frac{2\mathcal{N}_r}{\alpha_r}\right)X^+}(z,\bar{z}) \approx e^{-\frac{1}{2}\left(p_r^- - \frac{2\mathcal{N}_r}{\alpha_r}\right)(\rho+\bar{\rho})}(z,\bar{z})$$
$$\sim |z-Z_r|^{-2\left(p_r^+p_r^- - \mathcal{N}_r\right)} e^{-2\left(p_r^+p_r^- - \mathcal{N}_r\right)\left(\frac{\tau_0^{(r)}}{\alpha_r} - \mathrm{Re}\,\bar{N}_{00}^{rr}\right)} \tag{2.19}$$

for $z \sim Z_r$, where $\bar{N}_{00}^{rr}$ is defined in eq. (C.18). Using these, one can show that $V_r^{\mathrm{LC}} e^{-ip_r^+ X'^-}(Z_r,\bar{Z}_r)$ in eq. (2.10) subject to the on-shell condition (2.5) can be rewritten as

$$V_r^{\mathrm{LC}} e^{-ip_r^+ X'^-}(Z_r,\bar{Z}_r) \approx \alpha_r V_r^{\mathrm{DDF}}(Z_r,\bar{Z}_r) e^{2\mathrm{Re}\bar{N}_{00}^{rr}}\,, \tag{2.20}$$

where

$$V_r^{\mathrm{DDF}}(z,\bar{z}) = A_{-n_1}^{(r)i_1}\cdots \tilde{A}_{-\tilde{n}_1}^{(r)\tilde{\imath}_1}\cdots :e^{-ip_r^+X^- - i\left(p_r^- - \frac{2\mathcal{N}_r}{\alpha_r}\right)X^+ + ip_r^iX^i}(z,\bar{z}): \tag{2.21}$$

is the vertex operator corresponding to the DDF state.

Covariant expression. One subtle point to notice in eq. (2.14) is that the variables b, c should have zeros and poles at the zeros and poles of ∂X^+. Since the integration over X^- leads to the identification $X^+ \approx -\frac{i}{2}(\rho+\bar{\rho})$, the zeros and poles of ∂X^+ become those of $\partial\rho$, namely z_I and Z_r. Accordingly b should be inserted at z_I and c should be inserted at Z_r, if we wish to rewrite eq. (2.10) [7–10]. These insertions should come with appropriate factors made from $\rho, \bar{\rho}$ or X^+ in order that eq. (2.10) is reproduced. Using eqs. (2.20) and (C.25), one can show that eq. (2.10) can be rewritten as

$$F \sim \int \left[dX^\mu db dc d\tilde{b} d\tilde{c}\right] e^{-S_X - S_{bc}} \left|\sum_r \alpha_r Z_r\right|^2 \left(\lim_{z\to\infty}\frac{1}{|z|^4}c(z)\,\tilde{c}(\bar{z})\right)$$
$$\times \prod_I \left(\frac{b}{\partial^2\rho}(z_I)\frac{\tilde{b}}{\bar{\partial}^2\bar{\rho}}(\bar{z}_I)\right)\prod_{r=1}^N\left(c\tilde{c}V_r^{\mathrm{DDF}}\right)(Z_r,\bar{Z}_r)\ , \tag{2.22}$$

where S_X and S_{bc} are the free actions for $X^\mu = (X^\pm, X^i)$ and for b, c respectively.

One can show that eq. (2.22) yields the expression for the amplitudes in the conformal gauge. Using

$$\frac{b}{\partial^2\rho}(z_I) = \oint_{z_I}\frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\ ,$$

and deforming the contour, one can recast eq. (2.22) into the form

$$F \sim \int \left[dX^\mu db dc d\tilde{b} d\tilde{c}\right] e^{-S_X - S_{bc}}$$
$$\times \prod_I\left(\oint_{C_I}\frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\oint_{C_I}\frac{d\bar{z}}{2\pi i}\frac{\tilde{b}}{\bar{\partial}\bar{\rho}}(\bar{z})\right)\prod_{r=1}^N\left(c\tilde{c}V_r^{\mathrm{DDF}}\right)(Z_r,\bar{Z}_r)\ , \tag{2.23}$$

where the integration contour C_I is depicted in figure 1 in appendix C on the ρ-plane.

The amplitudes can be obtained by integrating the correlation function F over the $N-3$ moduli parameters $\mathcal{T}_I$ defined in eq. (C.4) as

$$\mathcal{A} \propto \int \prod_I d^2\mathcal{T}_I \, F(\mathcal{T}_I, \bar{\mathcal{T}}_I) \ . \tag{2.24}$$

One can see that the antighost insertion $\oint_{C_I} \frac{dz}{2\pi i} \frac{b}{\partial\rho}(z)$ corresponds to the quasiconformal vector field associated with the moduli parameter $\mathcal{T}_I$. The form of the amplitude is BRST invariant, because the vertex operator $c\tilde{c}V_r^{\mathrm{DDF}}$ is invariant and

$$\left\{ Q_{\mathrm{B}}, \oint_{C_I} \frac{dz}{2\pi i} \frac{b}{\partial\rho}(z) \right\} = \oint_{C_I} \frac{dz}{2\pi i} \frac{T}{\partial\rho}(z) \ , \tag{2.25}$$

yields a total derivative with respect to $\mathcal{T}_I$.

We can change the integration variables to Z_r $(r = 3, 4, \cdots, N-1)$ and

$$\mathcal{A} \propto \int \prod_{r=3}^{N-1} d^2 Z_r \left| \det \left(\frac{\partial \mathcal{T}_I}{\partial Z_r} \right) \right|^2 F(\mathcal{T}_I, \bar{\mathcal{T}}_I) \ . \tag{2.26}$$

With the expression (2.23) for F, the determinant factor can be combined with the antighost factors as

$$\begin{aligned}
\det\left(\frac{\partial \mathcal{T}_I}{\partial Z_r}\right) \times \prod_I \oint_{C_I} \frac{dz}{2\pi i}\frac{b}{\partial\rho}(z) &\propto \prod_{r=3}^{N-1} \left(\sum_I \frac{\partial \mathcal{T}_I}{\partial Z_r} \oint_{C_I} \frac{dz}{2\pi i} \frac{b}{\partial\rho}(z) \right) \\
&= \prod_{r=3}^{N-1} \left(\sum_I \oint_{C_I} \frac{dz}{2\pi i} \frac{\partial_{Z_r}(\rho(z) - \rho(z_I))}{\partial\rho(z)} b(z) \right. \\
&\qquad \left. - \sum_I \oint_{C_I} \frac{dz}{2\pi i} \frac{\partial_{Z_r}(\rho(z) - \rho(z_{I+1}))}{\partial\rho(z)} b(z) \right) \\
&\propto \prod_{r=3}^{N-1} \left(\sum_{s=1}^{N} \oint_{Z_s} \frac{dz}{2\pi i} \frac{\partial_{Z_r}(\rho(z) - \rho(z_{I^{(s)}}))}{\partial\rho(z)} b(z) \right) \ .
\end{aligned} \tag{2.27}$$

Performing the contour integrals around Z_s in the last line, we eventually obtain the familiar expression

$$\begin{aligned}
\mathcal{A} \propto & \int \left[dX^\mu db dc d\tilde{b} d\tilde{c} \right] e^{-S_X - S_{bc}} \\
& \times \prod_{s=1,2,N} \left(c\tilde{c} V_s^{\mathrm{DDF}} \right)(Z_s, \bar{Z}_s) \prod_{r=3}^{N-1} \int d^2 Z_r V_r^{\mathrm{DDF}}(Z_r, \bar{Z}_r) \ .
\end{aligned} \tag{2.28}$$

Hence the variables $X^\pm, b, c$ can be identified with those in the covariant formulation. From eq. (2.15), one can find the OPE's of the variables X^+, X'^-, b', c' to be

$$\begin{aligned}
\partial X^+(z) \partial X^+(z') &\sim \text{regular} \ , \\
\partial X^+(z) \partial X'^-(z') &\sim \frac{1}{(z-z')^2} \ , \\
\partial X'^-(z) \partial X'^-(z') &\sim -2\partial_z \partial_{z'} \left(\frac{1}{(z-z')^2} \frac{1}{\partial X^+(z) \partial X^+(z')} \right) \ , \\
b'(z) c'(z') &\sim \frac{1}{z-z'} \ ,
\end{aligned} \tag{2.29}$$

JHEP12(2009)010

and regular otherwise. These can also be deduced from the action (2.11). Using these OPE's, one can easily show that the energy-momentum tensor (2.13) satisfies the Virasoro algebra of central charge $c = 0$.

2.2 Noncritical case

The light-cone gauge string field theory in d ($d \neq 26$) space-time dimensions can be defined with the action given in appendix A. This time, the correlation functions we should consider is[3]

$$F \sim (2\pi)^2 \delta\left(\sum_{r=1}^{N} p_r^+\right) \delta\left(\sum_{r=1}^{N} p_r^-\right) \int \left[dX^i\right] e^{-S_{X^i} - \frac{d-2}{24}\Gamma[\phi]} \prod_r V_r^{\mathrm{LC}} \, . \tag{2.30}$$

We can follow the above procedure and introduce the variables $X^\pm, b, c$ without any problem. In this case, eq. (2.13) should be

$$T(z) = \partial X^+ \partial X'^- - \frac{1}{2}\partial X^i \partial X^i - \frac{d-2}{12}\left\{X^+, z\right\} - b'\partial c' \, . \tag{2.31}$$

By using $X^\pm, b, c$ defined in eq. (2.15), $T(z)$ can be rewritten as

$$T(z) = \partial X^+ \partial X^- - \frac{d-26}{12}\left\{X^+, z\right\} - \frac{1}{2}\partial X^i \partial X^i - 2b\partial c - \partial b c \, , \tag{2.32}$$

and the action $S_{X^\pm}$ should be

$$S_{X^\pm} = -\frac{1}{2\pi}\int d^2 z \left(\partial X^+ \bar{\partial} X^- + \bar{\partial} X^+ \partial X^-\right) + \frac{d-26}{24}\Gamma[\ln\left(-4\partial X^+ \bar{\partial} X^+\right)] \, . \tag{2.33}$$

Therefore in noncritical dimensions the worldsheet theory for $X^\pm$ is different from the usual free theory, and obviously the Lorentz symmetry is broken.

What we should study is this theory for $X^\pm$. It is a conformal field theory similar to the one for X^+ and X'^- in the previous subsection. We will study its properties in the next section.

Before closing this subsection, a comment is in order. For $d \neq 26$, one can define

$$\begin{aligned}
b'' &\equiv \left(\partial X^+\right)^\alpha b' \, , \\
c'' &\equiv \left(\partial X^+\right)^{-\alpha} c' \, , \\
X''^- &\equiv X'^- + \alpha \frac{b'c'}{\partial X^+} - \frac{\alpha}{2}\frac{\partial^2 X^+}{(\partial X^+)^2} + \alpha \frac{\tilde{b}'\tilde{c}'}{\bar{\partial} X^+} - \frac{\alpha}{2}\frac{\bar{\partial}^2 X^+}{(\bar{\partial} X^+)^2} \, ,
\end{aligned} \tag{2.34}$$

with

$$\alpha\left(\alpha + 1\right) = \frac{d-2}{12} \, . \tag{2.35}$$

[3] This time we have

$$\left[dX^i\right]_\phi \sim \left[dX^i\right] \mathrm{sgn}\left(\prod_{r=1}^{N} \alpha_r\right) e^{-\frac{d-2}{24}\Gamma[\phi]} \, ,$$

and we ignore the phase $\mathrm{sgn}\left(\prod_{r=1}^{N} \alpha_r\right)$ in the following.

Using these variables, the energy-momentum tensor (2.31) can be written as

$$T(z) = \partial X^+ \partial X''^- - \frac{1}{2}\partial X^i \partial X^i - b''\partial c'' - \alpha \partial \left(b'' c''\right) \ . \tag{2.36}$$

Therefore the worldsheet theory is a free theory with ghosts of noninteger spins. We may be able to study the theory using these variables, although we need to figure out the way to deal with the ghost zero-modes.

3 $X^\pm$ CFT

3.1 Action and correlation functions

The theory we would like to consider is with the action (2.33) and the energy-momentum tensor

$$T_{X^\pm}(z) = \partial X^+ \partial X^- - \frac{d-26}{12}\left\{X^+, z\right\} \ . \tag{3.1}$$

This theory will be well-defined if ∂X^+ has a nonvanishing expectation value. As in the case of the theory for X^+ and X'^-, we always consider this theory in the presence of the insertions of the vertex operators $e^{-ip_r^+ X^-}(Z_r, \bar{Z}_r)$ $(r = 1, \dots, N)$ with $\sum_{r=1}^N p_r^+ = 0$, so that the classical equation of motion implies

$$X^+(z, \bar{z}) = -i \sum_{r=1}^N p_r^+ \ln |z - Z_r|^2 = -\frac{i}{2}\left(\rho(z) + \bar{\rho}(\bar{z})\right) \ . \tag{3.2}$$

Thus the quantities we would like to calculate are the expectation values

$$\begin{aligned}\left\langle F\left[X^+, X^-\right] \prod_{r=1}^N e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right)\right\rangle \qquad\qquad&\\ \equiv \int \left[dX^+ dX^-\right] e^{-S_{X^\pm}} F\left[X^+, X^-\right] \prod_{r=1}^N e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right),&\end{aligned} \tag{3.3}$$

for functionals $F\left[X^+, X^-\right]$ which satisfy

$$F\left[X^+ + \epsilon_+, X^- + \epsilon_-\right] = F\left[X^+, X^-\right] , \tag{3.4}$$

for arbitrary constants $\epsilon_\pm$.

For functionals $F\left[X^+\right]$ which do not depend on X^-, it is formally possible to perform the path integral as in eq. (2.9) and obtain

$$\left\langle F\left[X^+\right] \prod_{r=1}^N e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right)\right\rangle \sim F\left[-\frac{i}{2}(\rho + \bar{\rho})\right] \exp\left(-\frac{d-26}{24}\Gamma\left[\ln\left(\partial\rho\bar{\partial}\bar{\rho}\right)\right]\right) , \tag{3.5}$$

up to the factor coming from the integration over the zero-modes of $X^\pm$. It is convenient to define

$$\left\langle F\left[X^+, X^-\right]\right\rangle_\rho \equiv \frac{\left\langle F\left[X^+, X^-\right] \prod_{r=1}^N e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right)\right\rangle}{\left\langle \prod_{r=1}^N e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right)\right\rangle} , \tag{3.6}$$

and we obtain

$$\left\langle F\left[X^+\right]\right\rangle_\rho = F\left[-\frac{i}{2}(\rho + \bar{\rho})\right] \ . \tag{3.7}$$

JHEP12(2009)010

$\boldsymbol{F\left[-\frac{i}{2}(\rho+\bar{\rho})\right] \exp\left(-\frac{d-26}{24}\Gamma\left[\ln\left(\partial\rho\bar{\partial}\bar{\rho}\right)\right]\right)}$ **as a generating functional.** The manipulation to obtain eq. (3.5) is rather formal because $\partial\rho(z)$ possesses zeros and poles and we need to specify the regularization procedure to define $\Gamma\left[\ln\left(\partial\rho\bar{\partial}\bar{\rho}\right)\right]$. As we mentioned in the last section, this was done in ref. [5] and $\Gamma\left[\ln\left(\partial\rho\bar{\partial}\bar{\rho}\right)\right]$ is given for arbitrary Mandelstam mapping ρ. Assuming that eq. (3.5) is true with Γ derived in ref. [5] (and also in appendix C), one can calculate correlation functions which involve X^-. Roughly speaking, differentiating the left hand side of eq. (3.5) with respect to p_N^+ and setting $p_N^+ = 0$, we obtain

$$\left\langle F\left[X^+\right] X^-(Z_N,\bar{Z}_N) \prod_{r=1}^{N-1} e^{-ip_r^+X^-}\left(Z_r,\bar{Z}_r\right)\right\rangle , \tag{3.8}$$

although the momentum conservation condition complicates the procedure a bit. In the same way, we can in principle obtain arbitrary correlation functions, treating the right hand side of eq. (3.5) as a kind of generating functional.

Since all the correlation functions on the complex plane are given this way, we define the theory based on eq. (3.5), rather than starting from the action (2.33). If one define the theory in this way, it is not a priori clear if $T_{X^\pm}(z)$ in eq. (3.1) can be considered as the energy-momentum tensor. In the rest of this section, we would like to calculate the correlation functions of $T_{X^\pm}(z)$ and examine if it can be regarded as the energy-momentum tensor of the theory.

One comment is in order. The operator $e^{-ip^+X^-}$ can be considered to create a hole of length α in the light-cone diagram. Therefore it is similar to the macroscopic observables in the old matrix models [11]. These operators are nonlocal objects and give rise to singularities at z_I where no operators are inserted. Taking $p^+ \to 0$ limit to obtain local operators is exactly what was done in the old matrix models.

3.2 Correlation functions of X^-

In order to calculate the correlation functions of $T_{X^\pm}(z)$, we need correlation functions of X^-. Let us follow the above-mentioned procedure and calculate them.

One point function. First we consider the simplest example, $\langle \partial X^-(z)\rangle_\rho$. In order to calculate this quantity we start from

$$\left\langle \prod_{r=0}^{N+1} e^{-ip_r^+X^-}\left(Z_r,\bar{Z}_r\right)\right\rangle \sim \exp\left(-\frac{d-26}{24}\Gamma\left[\ln\left(\partial\rho'\bar{\partial}\bar{\rho}'\right)\right]\right) , \tag{3.9}$$

where $p_{N+1}^+ = -p_0^+$ and

$$\rho'(z) = \sum_{r=0}^{N+1}\alpha_r \ln(z-Z_r) , \qquad \alpha_{N+1} = -\alpha_0 . \tag{3.10}$$

Then the one point function $\langle \partial X^-(z)\rangle_\rho$ can be given as

$$\langle \partial X^-(Z_0)\rangle_\rho = 2i\,\partial_{Z_0}\partial_{\alpha_0}\left(-\frac{d-26}{24}\Gamma\left[\ln\left(\partial\rho'\bar{\partial}\bar{\rho}'\right)\right]\right)\Bigg|_{\alpha_0=0} . \tag{3.11}$$

It is straightforward to evaluate the right hand side of eq. (3.11) by using the expression (C.22) for $\Gamma[\phi]$. In the calculations, one should notice the following points. While there are N interaction points $z'_{I'}$ for ρ', there are only $N-2$ interaction points z_I for $\rho = \lim_{\alpha_0 \to 0} \rho'$. Since

$$\partial\rho'(z) = \partial\rho(z) + \frac{\alpha_0}{z - Z_0} - \frac{\alpha_0}{z - Z_{N+1}}\,, \tag{3.12}$$

what happens in the limit $\alpha_0 \to 0$ is as follows: One of $z'_{I'}$'s, which we denote by $z'_{I(0)}$, goes to Z_0; One of $z'_{I'}$'s, which we denote by $z'_{I(N+1)}$, goes to Z_{N+1}; The other interaction points z'_I go to the interaction points z_I for ρ, which are denoted with the same subscripts. As we mentioned earlier, $e^{-ip_0^+ X^-}(Z_0)$ with finite p_0^+ should be considered as a nonlocal operator, which induces singularities at the interaction points where there are no operator insertions. $z'_{I(0)} \to Z_0$ in the limit $p_0^+ \to 0$ is consistent with the fact that $e^{-ip_0^+ X^-}(Z_0)$ tends to a local operator. One subtle point to notice is that if Z_0 is close to one of Z_r $(r = 1, \cdots, N)$ and the interaction point nearest to Z_r coincides with $z'_{I(0)}$, the Neumann coefficient $\bar{N}'^{rr}_{00}$ does not go to $\bar{N}^{rr}_{00}$ in the limit $p_0^+ \to 0$. Since $\Gamma\left[\ln\left(\partial\rho'\bar{\partial}\bar{\rho}'\right)\right]$ depends on $\bar{N}'^{rr}_{00}$, it implies that $e^{-ip_0^+ X^-}(Z_0)$ has a nonlocal effect even if we take $p_0^+ \to 0$ in such a configuration. Then we cannot expect to get correlation functions of local operators from $\Gamma\left[\ln\left(\partial\rho'\bar{\partial}\bar{\rho}'\right)\right]$. In the following, we will assume that Z_0 and Z_{N+1} are not close to any of Z_r $(r = 1, \cdots, N)$, to avoid such a situation. Namely we assume that $\rho(Z_0)$ and $\rho(Z_{N+1})$ are not in the regions of external propagators in the ρ plane. We calculate $\langle \partial X^-(Z_0)\rangle_\rho$ for such Z_0, and analytically continue the result to the whole complex plane.

Using the identities presented in appendix D, we obtain

$$\begin{aligned}
\left\langle \partial X^-(z)\right\rangle_\rho = \frac{d-26}{24}\,2i\Bigg[&-\sum_{r=1}^{N}\frac{1}{\alpha_r}\left(\frac{1}{z - z_I^{(r)}} - \frac{1}{z - Z_r}\right)\\
&-\sum_I \frac{1}{\partial^2\rho(z_I)}\frac{1}{(z - z_I)^2}\frac{\partial(-W)}{\partial z_I}\\
&-3\sum_I\left(\frac{1}{\partial^2\rho(z_I)}\frac{1}{(z - z_I)^3} - \frac{\partial^3\rho(z_I)}{2\left(\partial^2\rho(z_I)\right)^2}\frac{1}{(z - z_I)^2}\right)\Bigg],
\end{aligned} \tag{3.13}$$

where W is defined in eq. (C.15).

One can generalize eq. (3.11) and calculate the correlation functions with insertions of X^+ as

$$\begin{aligned}
&\left\langle \partial X^-(Z_0)\,F\left[X^+\right]\right\rangle_\rho\\
&\quad= \left.\frac{2i\partial_{Z_0}\partial_{\alpha_0}\left\langle F\left[X^+\right]\prod_{r=0}^{N+1} e^{-ip_r^+ X^-}(Z_r, \bar{Z}_r)\right\rangle}{\left\langle \prod_{r=0}^{N+1} e^{-ip_r^+ X^-}(Z_r, \bar{Z}_r)\right\rangle}\right|_{\alpha_0 = 0}\\
&\quad= \left\langle \partial X^-(Z_0)\right\rangle_\rho F\left[-\frac{i}{2}(\rho + \bar{\rho})\right] + \int d^2z\,\frac{1}{Z_0 - z}\left.\frac{\delta F\left[X^+\right]}{\delta X^+(z, \bar{z})}\right|_{X^+ = -\frac{i}{2}(\rho + \bar{\rho})}\,.
\end{aligned} \tag{3.14}$$

Since we are dealing with the correlation functions with source terms for X^-, we can read off the operator relations from these correlation functions. From eqs. (3.7) and (3.14), we obtain the OPE's

$$\partial X^+(z)\partial X^+(z') \sim \text{regular}\,, \qquad \partial X^-(z)\partial X^+(z') \sim \frac{1}{(z-z')^2}\,, \tag{3.15}$$

which are valid if z and z' are away from the singularities Z_r, z_I. These are consistent with eq. (2.29).

Two point function. The two point function for ∂X^- can be calculated by using eq. (3.13) as

$$\left\langle \partial X^-(z)\partial X^-(Z_0)\right\rangle_\rho = 2i\partial_{Z_0}\partial_{\alpha_0}\left\langle \partial X^-(z)\right\rangle_{\rho'}\Big|_{\alpha_0=0} + \left\langle \partial X^-(z)\right\rangle_\rho \left\langle \partial X^-(Z_0)\right\rangle_\rho\,. \tag{3.16}$$

Here we are interested in the singularity at $z = Z_0$. It is straightforward to calculate the right hand side of eq. (3.16) and obtain

$$\left\langle \partial X^-(z)\partial X^-(Z_0)\right\rangle_\rho = (2i)^2\left(-\frac{d-26}{12}\right)\partial_z\partial_{Z_0}\left[\frac{1}{(z-Z_0)^2}\frac{1}{\partial\rho(z)\partial\rho(Z_0)}\right] + \text{regular terms}\,. \tag{3.17}$$

From this, we deduce the OPE

$$\begin{aligned}
\partial X^-(z)\partial X^-(z') &\sim -\frac{d-26}{12}\partial_z\partial_{z'}\left[\frac{1}{(z-z')^2}\frac{1}{\partial X^+(z)\partial X^+(z')}\right]\\
&\sim -\frac{d-26}{12}\left[-\frac{1}{(z-z')^4}\frac{6}{(\partial X^+(z'))^2} - \frac{1}{(z-z')^3}3\partial\left(\frac{1}{(\partial X^+(z'))^2}\right)\right.\\
&\qquad\left. - \frac{1}{(z-z')^2}\frac{1}{2}\partial^2\left(\frac{1}{(\partial X^+(z'))^2}\right)\right]\,,
\end{aligned} \tag{3.18}$$

which is valid if z and z' are away from the singularities Z_r, z_I. This is consistent with eq. (2.29).

We can also obtain an expression for the correlation functions with X^+ insertions as we did in eq. (3.14):

$$\begin{aligned}
&\left\langle \partial X^-(z)\,\partial X^-(w)\,F\left[X^+\right]\right\rangle_\rho\\
&\quad= \left\langle \partial X^-(z)\,\partial X^-(w)\right\rangle_\rho F\left[-\frac{i}{2}(\rho+\bar\rho)\right]\\
&\qquad+\left\langle \partial X^-(z)\right\rangle_\rho \int d^2w'\,\frac{1}{w-w'}\,\frac{\delta F\left[X^+\right]}{\delta X^+(w',\bar w')}\Big|_{X^+=-\frac{i}{2}(\rho+\bar\rho)}\\
&\qquad+\left\langle \partial X^-(w)\right\rangle_\rho \int d^2z'\,\frac{1}{z-z'}\,\frac{\delta F\left[X^+\right]}{\delta X^+(z',\bar z')}\Big|_{X^+=-\frac{i}{2}(\rho+\bar\rho)}\\
&\qquad+\int d^2z'\int d^2w'\,\frac{1}{z-z'}\frac{1}{w-w'}\,\frac{\delta^2 F\left[X^+\right]}{\delta X^+(z',\bar z')\delta X^+(w',\bar w')}\Big|_{X^+=-\frac{i}{2}(\rho+\bar\rho)}\,.
\end{aligned} \tag{3.19}$$

3.3 Energy-momentum tensor

To be precise, the term $\partial X^+\partial X^-(z)$ in $T_{X^\pm}(z)$ given in eq. (3.1) is defined as

$$:\partial X^+\partial X^-(z): = \lim_{z'\to z}\left(\partial X^+(z')\partial X^-(z) - \frac{1}{(z'-z)^2}\right)\,. \tag{3.20}$$

Then the correlation functions with one $T_{X^\pm}(z)$ insertion can be evaluated as

$$\begin{aligned}&\left\langle T_{X^\pm}(z)F\left[X^+\right]\right\rangle_\rho\\&=\left\langle T_{X^\pm}(z)\right\rangle_\rho F\left[-\frac{i}{2}\left(\rho+\bar\rho\right)\right]-\frac{i}{2}\partial\rho\left(z\right)\int d^2z'\,\frac{1}{z-z'}\,\frac{\delta F\left[X^+\right]}{\delta X^+(z',\bar z')}\Bigg|_{X^+=-\frac{i}{2}(\rho+\bar\rho)}\,,\end{aligned}\tag{3.21}$$

where

$$\left\langle T_{X^\pm}(z)\right\rangle_\rho=-\frac{i}{2}\partial\rho\left(z\right)\left\langle\partial X^-\left(z\right)\right\rangle_\rho-\frac{d-26}{12}\left\{\rho,z\right\}\ .\tag{3.22}$$

One can evaluate the right hand side of eq. (3.22) and examine how it behaves around the possible singularities:

$$\left\langle T_{X^\pm}(z)\right\rangle_\rho\sim\begin{cases}\dfrac{1}{z-Z_r}\dfrac{\partial\left(-\frac{d-26}{24}\Gamma[\phi]\right)}{\partial Z_r} & z\sim Z_r\\ \text{regular} & z\sim z_I\\ \mathcal{O}\left(\frac{1}{z^4}\right) & z\sim\infty\end{cases}\ .\tag{3.23}$$

Eqs. (3.23) and (3.21) imply that $T_{X^\pm}(z)$ is regular at $z=z_I$ and ∞, if no operators are inserted there. Therefore, although $X^-(z)$ is singular at $z=z_I$ without any operator insertions, the energy momentum tensor is regular and conserved. This property is essential for constructing the BRST charge. From eq. (3.23) for $z\sim Z_r$ we can read off the OPE

$$T_{X^\pm}(z)e^{-ip_r^+X^-}(Z_r,\bar Z_r)\sim\frac{1}{z-Z_r}\partial e^{-ip_r^+X^-}(Z_r,\bar Z_r)\ .\tag{3.24}$$

Although $e^{-ip_r^+X^-}$ is a nonlocal operator, it behaves as a primary field of weight 0.

Using the OPE's (3.15) and (3.18), we can show that $T_{X^\pm}(z)$ satisfies

$$T_{X^\pm}(z)T_{X^\pm}(z')\sim\frac{\frac{1}{2}(28-d)}{(z-z')^4}+\frac{2}{(z-z')^2}T_{X^\pm}(z')+\frac{1}{z-z'}\partial T_{X^\pm}(z')\ .\tag{3.25}$$

Therefore the central charge of the Virasoro algebra in the $X^\pm$ CFT is $28-d$. We can also find that

$$T_{X^\pm}(z)\partial X^\pm(z')\sim\frac{1}{(z-z')^2}\partial X^\pm(z')+\frac{1}{z-z'}\partial^2X^\pm(z')\,,\tag{3.26}$$

and thus $\partial X^\pm$ are primary fields of weight 1.

4 BRST invariant formulation in noncritical dimensions

Since the worldsheet theory for $X^\pm$ is a CFT with Virasoro central charge $28-d$, with the transverse coordinates X^i added the total central charge of the system for $X^\pm$ and X^i is 26. Therefore with ghosts b and c, we can construct a nilpotent BRST charge Q_{B}.

As we have shown in section 2, the amplitude for the light-cone gauge string field theory can be rewritten by using these variables. We start from the correlation function given in eq. (2.30), where the vertex operator V_r^{LC} is of the form (2.4) but with the on-shell condition

$$\frac{1}{2}\left(-2p_r^+p_r^- + p_r^ip_r^i\right) + \mathcal{N}_r = \frac{d-2}{24}\,. \tag{4.1}$$

We can proceed in the same way as in section 2 and eventually obtain

$$\begin{aligned} F \sim \int \left[dX^{\pm}dX^i dbdcd\tilde{b}d\tilde{c}\right] e^{-S_{X^i}-S_{X^{\pm}}-S_{bc}} \\ \times \prod_I \left(\oint_{C_I}\frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\oint_{C_I}\frac{d\bar{z}}{2\pi i}\frac{\tilde{b}}{\bar{\partial}\bar{\rho}}(\bar{z})\right) \\ \times \prod_{r=1}^N \left(c\tilde{c}V_r^{\mathrm{DDF}}\exp\left(-i\frac{d-26}{24}\frac{X^+}{p_r^+}\right)\right)(Z_r,\bar{Z}_r) \\ \times \prod_{r=1}^N \exp\left(i\frac{d-26}{24}\frac{X^+}{p_r^+}\right)\left(z_I^{(r)},\bar{z}_I^{(r)}\right)\,, \end{aligned} \tag{4.2}$$

where V_r^{DDF} is defined in the same way as in eq. (2.21).

It is easy to show that $V_r^{\mathrm{DDF}}\exp\left(-i\frac{d-26}{24}\frac{X^+}{p_r^+}\right)$ is a primary field of weight $(1,1)$ and $T_{X^{\pm}}(z)$ is regular even with the insertion $\exp\left(i\frac{d-26}{24}\frac{X^+}{p_r^+}\right)(z_I^{(r)},\bar{z}_I^{(r)})$. Therefore eq. (4.2) gives a BRST invariant expression for the amplitude. $V_r^{\mathrm{DDF}}\exp\left(-i\frac{d-26}{24}\frac{X^+}{p_r^+}\right)$ may look as a vertex operator with momentum

$$p_r'^- = p_r^- + \frac{1}{p_r^+}\frac{d-26}{24}\,, \tag{4.3}$$

instead of p_r^-, but the momentum which is conserved is p^-. The conserved momentum can be identified with the operator

$$\oint\frac{dz}{2\pi i}i\partial X^-(z)\,, \tag{4.4}$$

on the worldsheet, which is conserved at the interaction points with insertions $\exp\left(i\frac{d-26}{24}\frac{X^+}{p_r^+}\right)$.

5 Discussions

In this paper, we have constructed a BRST invariant worldsheet theory which corresponds to the light-cone gauge string field theory in d ($d \neq 26$) space-time dimensions. The worldsheet theory for the longitudinal coordinate variables $X^{\pm}$ is different from the usual free theory, but it is a CFT with $c = 28 - d$. Our results provide yet another way to construct string theories in noncritical dimensions. The BRST invariant formulation will be useful to study D-branes for such string theories.

Now that the CFT is given, we can at least formally construct the interaction vertices of the string field theory based on this CFT through the prescription of ref. [12]. Since we have constructed the CFT on the worldsheet of the light-cone string diagram, the gauge unfixed version of the string field theory is supposed to possess the joining-splitting type of interactions. Such a theory looks similar to the $\alpha = p^+$ HIKKO theory given in ref. [13].

The results in this paper should be generalized to be used in regularizing string field theory. One should consider the $X^{\pm}$ on the Riemann surfaces with higher genera, in order to check if it works for regularizing the UV and IR divergences. One should also construct a supersymmetric version of the CFT. The light-cone gauge superstring field theory in noncritical dimensions can be used to dimensionally regularize the tree amplitudes of the critical theory, as was discussed [4]. It is possible to generalize the calculations performed in this paper into the superstring case, although they are much more complicated. We will present these results elsewhere.

Acknowledgments

This work was supported in part by Grant-in-Aid for Scientific Research (C) (20540247) and Grant-in-Aid for Young Scientists (B) (19740164) from the Ministry of Education, Culture, Sports, Science and Technology (MEXT).

A Action of light-cone gauge string field theory

In order to fix the notation, we present the action for the light-cone gauge bosonic string field theory in d space-time dimensions. The worldsheet variables are X^i $(i = 1, \cdots, d-2)$. The action of the string field theory takes the form

$$S = \int dt \left[\frac{1}{2} \int d1d2 \, \langle R(1,2)\, |\Phi\rangle_1 \left(i\frac{\partial}{\partial t} - \frac{L_0^{\mathrm{LC}(2)} + \tilde{L}_0^{\mathrm{LC}(2)} - \frac{d-2}{12}}{\alpha_2} \right) |\Phi\rangle_2 \right.$$
$$\left. + \frac{2g}{3} \int d1d2d3 \, \langle V_3(1,2,3)|\, \Phi\rangle_1 |\Phi\rangle_2 |\Phi\rangle_3 \right] . \qquad \text{(A.1)}$$

Here $L_0^{\mathrm{LC}(r)}$ is the zero-mode of the transverse Virasoro generators for the r-th string, g is the coupling constant, and dr denotes the integration measure for the momentum zero-modes of the r-th string defined as

$$dr = \frac{\alpha_r d\alpha_r}{4\pi} \frac{d^{d-2} p_r}{(2\pi)^{d-2}} , \qquad \text{(A.2)}$$

where $\alpha_r = 2p_r^+$ is the string-length parameter of the r-th string. $\langle R(1,2)|$ is the reflector given by

$$\begin{aligned} \langle R(1,2)| &= \delta(1,2) \frac{1}{\alpha_1} \, {}_2\langle 0| \, {}_1\langle 0| e^{-\sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_n^{i(1)} \alpha_n^{i(2)} + \tilde{\alpha}_n^{i(1)} \tilde{\alpha}_n^{i(2)} \right)} , \\ \delta(1,2) &= 4\pi\delta\left(\alpha_1 + \alpha_2\right) (2\pi)^{d-2} \delta^{d-2}\left(p_1 + p_2\right) . \end{aligned} \qquad \text{(A.3)}$$

$\langle V_3(1,2,3)|$ denotes the three-string interaction vertex defined as

$$\langle V_3(1,2,3)| = 4\pi\delta\left(\sum_{r=1}^{3} \alpha_r \right) \mathrm{sgn}(\alpha_1\alpha_2\alpha_3) \left| \frac{e^{-2\hat{\tau}_0 \sum_{r=1}^{3} \frac{1}{\alpha_r}}}{\alpha_1\alpha_2\alpha_3} \right|^{\frac{d-2}{24}} \left\langle V_3^{\mathrm{LPP}}(1,2,3) \right| , \qquad \text{(A.4)}$$

Here $\left\langle V_3^{\mathrm{LPP}}(1,2,3) \right|$ is the three-string LPP vertex [12] for the transverse coordinate variables X^i and $\hat{\tau}_0 = \sum_{r=1}^{3} \alpha_r \ln|\alpha_r|$. The string field $|\Phi\rangle$ is taken to obey the reality and the level-matching conditions.

B Mandelstam mapping

Let ρ be the standard complex coordinate on the N-string tree diagram with the joining-splitting type interaction. The portion on the ρ-plane corresponding to the r-th external string ($r = 1, \ldots, N$) is mapped to the unit disk, $|w_r| \leq 1$, of the r-th string as

$$\rho = \alpha_r \ln w_r + \tau_0^{(r)} + i\beta_r \,, \tag{B.1}$$

where $\tau_0^{(r)} + i\beta_r$ is the coordinate on the ρ-plane at which the r-th string interacts.

The N-string tree diagram is mapped to the complex z-plane with N punctures by the Mandelstam mapping [2]

$$\rho(z) = \sum_{r=1}^{N} \alpha_r \ln(z - Z_r) \,, \qquad \sum_{r=1}^{N} \alpha_r = 0 \,, \tag{B.2}$$

where the puncture $z = Z_r$ corresponds to the origin of the unit disk $w_r = 0$. The $N-2$ interaction points z_I ($I = 1, \cdots, N-2$) are determined by $\partial\rho(z_I) = 0$. These are related to the interaction points on the ρ-plane by

$$\rho(z_I^{(r)}) = \tau_0^{(r)} + i\beta_r \,, \tag{B.3}$$

where $z_I^{(r)}$ denotes one of z_I at which the r-th external string interacts. The fact that z_I's are the zeros of $\partial\rho(z)$ yields

$$\partial\rho(z) = \left(\sum_{s=1}^{N} \alpha_s Z_s\right) \frac{\prod_I (z - z_I)}{\prod_{r=1}^{N} (z - Z_r)} \,, \tag{B.4}$$

$$\partial^2\rho(z_I) = \left(\sum_{s=1}^{N} \alpha_s Z_s\right) \frac{\prod_{J \neq I} (z_I - z_J)}{\prod_{r=1}^{N} (z_I - Z_r)} \,, \tag{B.5}$$

$$\frac{\sum_{r=1}^{N} \alpha_r Z_r^2}{\sum_{s=1}^{N} \alpha_s Z_s} = -\left(\sum_I z_I - \sum_{r=1}^{N} Z_r\right) \,, \tag{B.6}$$

$$\alpha_r = \left(\sum_{s=1}^{N} \alpha_s Z_s\right) \frac{\prod_I (Z_r - z_I)}{\prod_{s \neq r} (Z_r - Z_s)} \,. \tag{B.7}$$

C Computation of $\Gamma[\phi]$

$\Gamma[\phi]$ can be obtained by evaluating the Liouville action for the metric (2.1) on the z-plane [5, 6]. Here we will present an alternative derivation. $e^{-\Gamma[\phi]}$ can be regarded as the partition function of the light-cone gauge string theory in 26 space-time dimensions from the view point of the worldsheet CFT on the light-cone diagram. Therefore the variation $\delta\Gamma[\phi]$ under

$$Z_r \to Z_r + \delta Z_r \,, \quad \bar{Z}_r \to \bar{Z}_r + \delta\bar{Z}_r \,, \quad \alpha_r \to \alpha_r + \delta\alpha_r \qquad (r = 1, \ldots, N) \,, \tag{C.1}$$

can be given by using the expectation value of the transverse energy-momentum tensor T_{X^i}. We can obtain $\Gamma[\phi]$ by integrating $\delta\Gamma[\phi]$. The result should be up to a factor which does not change under the variation eq. (C.1). Such a factor can be fixed by imposing the factorization condition in each case [4].

Before we begin the calculation, two comments are in order. Because of the constraint $\sum_{r=1}^{N} \alpha_r = 0$, all the $\delta\alpha_r$'s cannot be treated as independent variations. Here we think of $\delta\alpha_r$ with $r = 1, \cdots, N-1$ as independent of each other and $\delta\alpha_N$ as being determined by the relation

$$\delta\alpha_N = -\sum_{r=1}^{N-1} \delta\alpha_r \ . \tag{C.2}$$

We also note that $\Gamma[\phi]$ involves divergence originating from the infinite length of the external lines. We regularize it by cutting off the r-th external line so that its length becomes $\alpha_r \ell_r$, where ℓ_r is a large positive constant. By doing so, the contribution from the r-th external line to the partition function becomes $e^{-2\ell_r}$, and the divergences of $\Gamma[\phi]$ become independent of the parameters α_r, Z_r. On the z-plane, this corresponds to cutting a hole of radius ϵ_r around Z_r, where ϵ_r and ℓ_r are related as

$$\alpha_r \ell_r = \mathrm{Re}\left(\rho(z_I^{(r)}) - \rho(Z_r + \epsilon_r)\right) \ . \tag{C.3}$$

Variation $\boldsymbol{\delta\left(-\Gamma[\phi]\right)}$. The variations (C.1) correspond to the variations of the following parameters of the light-cone diagram: (I) the moduli parameters

$$\mathcal{T}_I \equiv \rho\left(z_{I+1}\right) - \rho\left(z_I\right) \ ; \tag{C.4}$$

(II) the heights $\alpha_r \ell_r$ of the external cylinders; (III) the circumferences $2\pi\alpha_r$ of the external cylinders. Thus the variation $\delta(-\Gamma[\phi])$ is expressed as

$$\delta\left(-\Gamma[\phi]\right) = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) \, , \tag{C.5}$$

where

$$\begin{aligned}
(\mathrm{I}) &= \sum_I \delta\mathcal{T}_I \oint_{C_I} \frac{d\rho}{2\pi i} \left\langle T_{X^i}(\rho) \right\rangle + \mathrm{c.c.} \, , \\
(\mathrm{II}) &= \sum_{r=1}^{N-1} \left[-\ell_r \delta\alpha_r \oint_{\rho(Z_r)} \frac{d\rho}{2\pi i} \left\langle T_{X^i}(\rho) \right\rangle + \ell_N \delta\alpha_r \oint_{\rho(Z_N)} \frac{d\rho}{2\pi i} \left\langle T_{X^i}(\rho) \right\rangle \right] + \mathrm{c.c.} \, , \\
(\mathrm{III}) &= \sum_{r=1}^{N-1} i2\pi\delta\alpha_r \int_{L_{rN}} \frac{d\rho}{2\pi i} \left\langle T_{X^i}(\rho) \right\rangle + \mathrm{c.c.} \ .
\end{aligned} \tag{C.6}$$

Here c.c. stands for the complex conjugate. The integration contour C_I of the term (I) lies between the consecutive interaction points $\rho(z_{I+1})$ and $\rho(z_I)$ as depicted in figure 1. The integration path L_{rN} of term (III) in eq. (C.6) is a line stretching from the asymptotic region of the r-th external string to that of the N-th string on the ρ-plane. As an example, the path L_{1N} is depicted in figure 1. On the z-plane, the path L_{rN} becomes a segment connecting the two punctures Z_r and Z_N with the orientation from Z_r to Z_N.

The expectation value of $T_{X^i}(\rho)$ can be evaluated by going to the z-plane. Since

$$T_{X^i}(\rho) = \frac{1}{(\partial\rho(z))^2}\left(T_{X^i}(z) - 2\{\rho, z\}\right) \ , \qquad \left\langle T_{X^i}(z) \right\rangle = 0 \, , \tag{C.7}$$

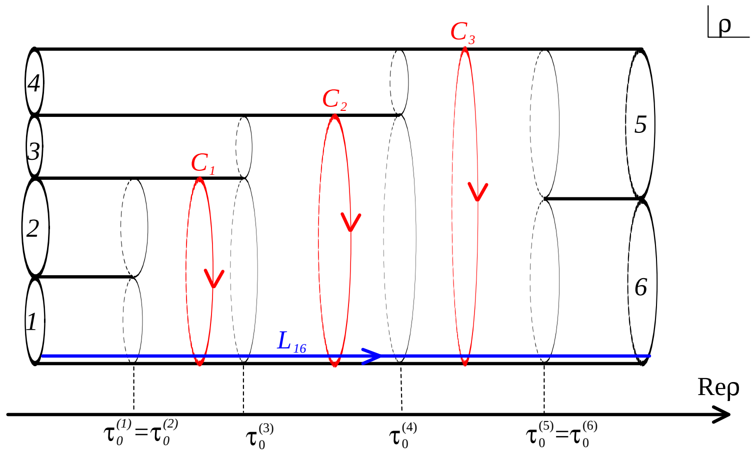

Figure 1. A typical N-string tree diagram with $N = 6$. The contours C_I and the path L_{1N} on the ρ-plane of the integrals in eq. (C.6) are depicted for this case.

we obtain

$$\langle T_{X^i}(\rho)\rangle = \frac{1}{(\partial\rho(z))^2}\left(-2\{\rho, z\}\right) \ . \tag{C.8}$$

Using eq. (C.8), one can rearrange the term (I) in eq. (C.6) as

$$\begin{aligned}
&\sum_I \delta T_I \oint_{C_I} \frac{d\rho}{2\pi i} \langle T_{X^i}(\rho)\rangle \\
&\qquad = \sum_I \delta\left(\rho(z_{I+1}) - \rho(z_I)\right) \oint_{C_I} \frac{dz}{2\pi i} \frac{(-2\{\rho, z\})}{\partial\rho(z)} \\
&\qquad = \sum_{r=1}^{N} \delta\rho(z_I^{(r)}) \oint_{Z_r} \frac{dz}{2\pi i} \frac{(-2\{\rho, z\})}{\partial\rho(z)} + \sum_I \delta\rho(z_I) \oint_{z_I} \frac{dz}{2\pi i} \frac{(-2\{\rho, z\})}{\partial\rho(z)} \\
&\qquad\quad + \delta\rho(z_I^{\infty}) \oint_{\infty} \frac{dz}{2\pi i} \frac{(-2\{\rho, z\})}{\partial\rho(z)} \,,
\end{aligned} \tag{C.9}$$

where z_I^{∞} denotes the interaction point closest to $z = \infty$.

From eq. (C.2), one can easily find that term (II) becomes

$$\text{(II)} = -\sum_{r=1}^{N} \ell_r \delta\alpha_r \oint_{Z_r} \frac{dz}{2\pi i} \frac{(-2\{\rho, z\})}{\partial\rho(z)} + \text{c.c.} \ . \tag{C.10}$$

We can recast the r-th term of (III) in eq. (C.6) into

$$\begin{aligned}
i2\pi\delta\alpha_r \int_{L_{rN}} \frac{d\rho}{2\pi i} \langle T_{X^i}(\rho)\rangle &= i2\pi\delta\alpha_r \int_{L_{rN}} \frac{dz}{2\pi i} \frac{(-2\{\rho, z\})}{\partial\rho(z)} \\
&= i2\pi\delta\alpha_r \int_{\tilde{L}_{rN}} \frac{dz}{2\pi i} \frac{\ln(z-Z_r)-\ln(z-Z_N)}{2\pi i} \frac{(-2\{\rho, z\})}{\partial\rho(z)} \ .
\end{aligned} \tag{C.11}$$

Here we have taken the cut of the function $\ln(z - Z_r) - \ln(z - Z_N)$ in the integrand on the right hand side to be the segment L_{rN} and the integration path $\tilde{L}_{rN}$ to be the sum

JHEP12(2009)010

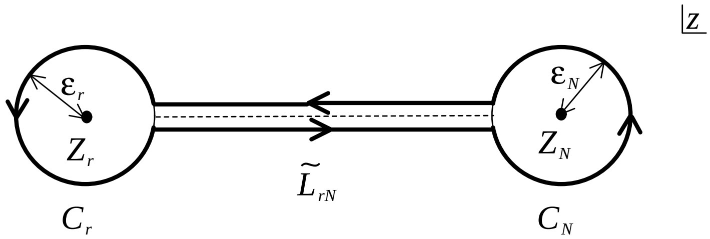

Figure 2. The integration path $\tilde{L}_{rN}$ consists of the two oriented segments described by the arrowed bold lines. The dashed line denotes the cut of the integrand on the right hand side of eq. (C.11).

of the two oriented segments connecting two punctures Z_r and Z_N as depicted in figure 2. Deforming the contours to rearrange eq. (C.11) further and summing over r, we obtain

$$i2\pi \sum_{r=1}^{N} \delta\alpha_r \int_{L_{rN}} \frac{d\rho}{2\pi i} \langle T_{X^i}(\rho)\rangle = -\left(\sum_{r=1}^{N}\int_{C_r} + \sum_I \oint_{z_I} + \oint_\infty\right) \frac{dz}{2\pi i}\delta\rho(z)\frac{(-2\{\rho,z\})}{\partial\rho(z)}\,, \tag{C.12}$$

where C_r and C_N are depicted in figure 2.

Eventually we obtain the expression

$$\begin{aligned}\delta\left(-\Gamma[\phi]\right) = \Bigg[&-\sum_{r=1}^{N}\int_{C_r}\frac{dz}{2\pi i}\frac{\delta\rho(z)-\delta\rho(z_I^{(r)})+\delta\alpha_r\ell_r}{\partial\rho(z)}\left(-2\{\rho,z\}\right)\\ &-\sum_I \oint_{z_I}\frac{dz}{2\pi i}\frac{\delta\rho(z)-\delta\rho(z_I)}{\partial\rho(z)}\left(-2\{\rho,z\}\right)\\ &-\oint_\infty \frac{dz}{2\pi i}\frac{\delta\rho(z)-\delta\rho(z_I^\infty)}{\partial\rho(z)}\left(-2\{\rho,z\}\right)\Bigg] + \text{c.c.}\,.\end{aligned} \tag{C.13}$$

Evaluation of the contour integrals. Let us evaluate the contour integrals on the right hand side of eq. (C.13). For $z \sim Z_r, z_I, \infty$, the Schwarzian derivative $\{\rho, z\}$ behaves as

$$-2\{\rho,z\} = \begin{cases} -\dfrac{1}{(z-Z_r)^2} + \dfrac{1}{z-Z_r}\dfrac{\partial(-W)}{\partial Z_r} + \mathcal{O}\left((z-Z_r)^0\right) & z\sim Z_r \\ \dfrac{3}{(z-z_I)^2} + \dfrac{1}{z-z_I}\dfrac{\partial(-W)}{\partial z_I} + \mathcal{O}\left((z-z_I)^0\right) & z\sim z_I \\ \mathcal{O}\left(\frac{1}{z^4}\right) & z\sim\infty \end{cases}, \tag{C.14}$$

where W is a function of $Z_r, \bar{Z}_r, z_I, \bar{z}_I$ defined as

$$W(Z_r,\bar{Z}_r,z_I,\bar{z}_I) \equiv -2\left(\sum_{I>J}\ln|z_I-z_J|^2 + \sum_{r>s}\ln|Z_r-Z_s|^2 - \sum_{I,r}\ln|Z_r-z_I|^2\right)\,. \tag{C.15}$$

For the contour integral along C_r, one should notice that $\delta\rho(z)$ involves a term

$\delta\alpha_r \ln(z - Z_r)$. This part can be integrated as

$$\int_{C_r} \frac{dz}{2\pi i} \frac{\delta\alpha_r \ln(z - Z_r)}{\partial\rho(z)} (-2\{\rho, z\}) \sim -\frac{1}{\alpha_r} \int_{C_r} \frac{dz}{2\pi i} \frac{\delta\alpha_r \ln(z - Z_r)}{z - Z_r}$$
$$= -\frac{\delta\alpha_r}{\alpha_r} \int \frac{d\theta}{2\pi} (\ln \epsilon_r + i\theta)$$
$$= -\frac{\delta\alpha_r}{\alpha_r} (\ln \epsilon_r + \text{imaginary part}) \, , \tag{C.16}$$

for $\epsilon_r \sim 0$. The other terms can be evaluated by taking the residues of the simple poles and one can show

$$-\sum_{r=1}^{N} \int_{C_r} \frac{dz}{2\pi i} \frac{\delta\rho(z) - \delta\rho(z_I^{(r)}) + \delta\alpha_r \ell_r}{\partial\rho(z)} (-2\{\rho, z\}) + \text{c.c}$$
$$= \delta \left(-2 \sum_{r=1}^{N} \text{Re} \bar{N}_{00}^{rr} \right) + \sum_{r=1}^{N} \left(\delta Z_r \frac{\partial\,(-W)}{\partial Z_r} + \text{c.c.} \right) \, , \tag{C.17}$$

where $\bar{N}_{00}^{rr}$ is a Neumann coefficient given by

$$\bar{N}_{00}^{rr} = -\sum_{s \neq r} \frac{\alpha_s}{\alpha_r} \ln(Z_r - Z_s) + \frac{\tau_0^{(r)} + i\beta_r}{\alpha_r} \, , \tag{C.18}$$

and $\tau_0^{(r)} + i\beta_r$ is defined in eq. (B.3).

For $z \sim z_I$, one can show

$$\frac{\delta\rho(z) - \delta\rho(z_I)}{\partial\rho(z)} = -\delta z_I + (z - z_I)\delta \left(\frac{1}{2} \ln \partial^2\rho(z_I) \right) + \mathcal{O}\left((z - z_I)^2 \right) \, , \tag{C.19}$$

using

$$\delta z_I = -\frac{\partial\delta\rho(z_I)}{\partial^2\rho(z_I)} \, , \tag{C.20}$$

which is derived by varying the equation $\partial\rho(z_I) = 0$ under eq. (C.1). The contour integral around z_I can be obtained as

$$-\sum_I \oint_{z_I} \frac{dz}{2\pi i} \frac{\delta\rho(z) - \delta\rho(z_I)}{\partial\rho(z)} (-2\{\rho, z\}) + \text{c.c.}$$
$$= \delta \left(-3 \sum_I \ln \left| \partial^2\rho(z_I) \right| \right) + \sum_I \left(\delta z_I \frac{\partial\,(-W)}{\partial z_I} + \text{c.c.} \right) \, . \tag{C.21}$$

It is easy to see that the integral around ∞ vanishes.

Putting all the pieces together, we obtain

$$-\Gamma[\phi] = -W - 2 \sum_{r=1}^{N} \text{Re}\, \bar{N}_{00}^{rr} - 3 \sum_I \ln \left| \partial^2\rho(z_I) \right| \, . \tag{C.22}$$

This form of $\Gamma[\phi]$ is useful in the calculations in section 3.

Other expressions. We note that W can be described as

$$W = 2\sum_{r=1}^{N} \ln|\alpha_r| - 4\ln\left|\sum_{r=1}^{N}\alpha_r Z_r\right| - 2\sum_I \ln\left|\partial^2\rho(z_I)\right| , \tag{C.23}$$

which follows from eqs. (B.5) and (B.7). This yields

$$e^{-\Gamma[\phi]} = \prod_{r=1}^{N}|\alpha_r|^{-2}\left|\sum_{s=1}^{N}\alpha_s Z_s\right|^4 e^{-2\sum_{r=1}^{N}\mathrm{Re}\,\bar{N}_{00}^{rr}}\prod_I\left|\partial^2\rho(z_I)\right|^{-1} . \tag{C.24}$$

This expression is the one obtained by the method in ref. [5]. We also note that

$$e^{-\Gamma[\phi]} = \prod_{r=1}^{N}|\alpha_r|^{-1} e^{-\frac{1}{2}W}\left|\sum_{s=1}^{N}\alpha_s Z_s\right|^2 e^{-2\sum_{r=1}^{N}\mathrm{Re}\,\bar{N}_{00}^{rr}}\prod_I\left|\partial^2\rho(z_I)\right|^{-2} , \tag{C.25}$$

which is used in sections 2 and 4.

D Correlation functions of $\partial X^-(z)$'s

In this appendix, we present the details of the calculations to obtain eqs. (3.13) and (3.17).

Let us consider the interaction points $z'_I, z'_{I(0)}$ and $z'_{I(N+1)}$ for $\rho'(z)$ defined in eq. (3.12), which tend to z_I, Z_0 and Z_{N+1} as $\alpha_0 \to 0$ respectively. From eq. (3.12), we can obtain the expansion of $z'_I, z'_{I(0)}$ and $z'_{I(N+1)}$ in terms of α_0,

$$\begin{aligned}
z'_I - z_I &= -\frac{\alpha_0}{\partial^2\rho(z_I)}\left(\frac{1}{z_I - Z_0} - \frac{1}{z_I - Z_{N+1}}\right) + \mathcal{O}\left(\alpha_0^2\right) , \\
z'_{I(0)} - Z_0 &= -\frac{\alpha_0}{\partial\rho(Z_0)}\left[1 + \left(\frac{\partial^2\rho(Z_0)}{(\partial\rho(Z_0))^2} + \frac{1}{\partial\rho(Z_0)}\frac{1}{Z_0 - Z_{N+1}}\right)\alpha_0 + \mathcal{O}\left(\alpha_0^2\right)\right] , \\
z'_{I(N+1)} - Z_{N+1} &= \frac{\alpha_0}{\partial\rho(Z_{N+1})}\left[1 - \left(\frac{\partial^2\rho(Z_{N+1})}{(\partial\rho(Z_{N+1}))^2} + \frac{1}{\partial\rho(Z_{N+1})}\frac{1}{Z_{N+1} - Z_0}\right)\alpha_0 + \mathcal{O}\left(\alpha_0^2\right)\right] .
\end{aligned} \tag{D.1}$$

The Neumann coefficients $\bar{N}'^{rr}_{00}$ for the Mandelstam mapping ρ' behave as

$$\begin{aligned}
\mathrm{Re}\,\bar{N}'^{rr}_{00} &= \mathrm{Re}\,\bar{N}^{rr}_{00} + \frac{\alpha_0}{\alpha_r}\ln\left|\frac{\left(z_I^{(r)} - Z_0\right)(Z_r - Z_{N+1})}{(Z_r - Z_0)\left(z_I^{(r)} - Z_{N+1}\right)}\right| + \mathcal{O}\left(\alpha_0^2\right) \qquad (r \neq 0, N+1) , \\
\mathrm{Re}\,\bar{N}'^{00}_{00} &= \ln\left|\frac{\alpha_0}{\partial\rho(Z_0)}\right| - 1 \\
&\quad + \mathrm{Re}\left(\frac{1}{2}\frac{\partial^2\rho(Z_0)}{(\partial\rho(Z_0))^2} + \frac{1}{\partial\rho(Z_0)(Z_0 - Z_{N+1})}\right)\alpha_0 + \mathcal{O}\left(\alpha_0^2\right) , \\
\mathrm{Re}\,\bar{N}'^{N+1N+1}_{0\ 0} &= \ln\left|\frac{\alpha_0}{\partial\rho(Z_{N+1})}\right| - 1 \\
&\quad - \mathrm{Re}\left(\frac{1}{2}\frac{\partial^2\rho(Z_{N+1})}{(\partial\rho(Z_{N+1}))^2} + \frac{1}{\partial\rho(Z_{N+1})(Z_{N+1} - Z_0)}\right)\alpha_0 + \mathcal{O}\left(\alpha_0^2\right) .
\end{aligned} \tag{D.2}$$

Using eqs. (B.4) and (D.1), we obtain

$$
\begin{aligned}
W' = {} & 8\ln|\alpha_0| + W - 4\ln|\partial\rho(Z_0)\partial\rho(Z_{N+1})| \\
& + 2\,\mathrm{Re}\Bigg[-\sum_I \frac{1}{\partial^2\rho(z_I)}\left(\frac{1}{z_I - Z_0} - \frac{1}{z_I - Z_{N+1}}\right)\frac{\partial W}{\partial z_I} \\
& \qquad + 4\frac{\partial^2\rho(Z_0)}{(\partial\rho(Z_0))^2} - 4\frac{\partial^2\rho(Z_{N+1})}{(\partial\rho(Z_{N+1}))^2} \\
& \qquad + 2\frac{1}{\partial\rho(Z_0)}\frac{1}{Z_0 - Z_{N+1}} - 2\frac{1}{\partial\rho(Z_{N+1})}\frac{1}{Z_{N+1} - Z_0}\Bigg]\alpha_0 + \mathcal{O}\left(\alpha_0^2\right), \qquad \text{(D.3)}
\end{aligned}
$$

and

$$
\begin{aligned}
\partial^2\rho'\left(z'_I\right) = {} & \partial^2\rho\left(z_I\right) + \left(-\frac{1}{\left(z_I - Z_0\right)^2} + \frac{1}{\left(z_I - Z_{N+1}\right)^2}\right)\alpha_0 \\
& - \frac{\partial^3\rho\left(z_I\right)}{\partial^2\rho\left(z_I\right)}\left(\frac{1}{z_I - Z_0} - \frac{1}{z_I - Z_{N+1}}\right)\alpha_0 + \mathcal{O}\left(\alpha_0^2\right), \\
\partial^2\rho'\left(z'_{I^{(0)}}\right) = {} & -\frac{\left(\partial\rho\left(Z_0\right)\right)^2}{\alpha_0} \\
& \times\left[1 - \left(\frac{3\partial^2\rho\left(Z_0\right)}{\left(\partial\rho\left(Z_0\right)\right)^2} + \frac{2}{\partial\rho\left(Z_0\right)\left(Z_0 - Z_{N+1}\right)}\right)\alpha_0 + \mathcal{O}\left(\alpha_0^2\right)\right], \\
\partial^2\rho'\left(z'_{I^{(N+1)}}\right) = {} & \frac{\left(\partial\rho\left(Z_{N+1}\right)\right)^2}{\alpha_0} \\
& \times\left[1 + \left(\frac{3\partial^2\rho\left(Z_{N+1}\right)}{\left(\partial\rho\left(Z_{N+1}\right)\right)^2} + \frac{2}{\partial\rho\left(Z_{N+1}\right)\left(Z_{N+1} - Z_0\right)}\right)\alpha_0 + \mathcal{O}\left(\alpha_0^2\right)\right]. \qquad \text{(D.4)}
\end{aligned}
$$

Gathering all the relations obtained above, we have

$$
\begin{aligned}
\Gamma[\ln\left(\partial\rho'\bar{\partial}\bar{\rho}'\right)] = {} & 6\ln|\alpha_0| - 4 + \Gamma[\ln\left(\partial\rho\bar{\partial}\bar{\rho}\right)] \\
& + 2\,\mathrm{Re}\Bigg[\sum_{r=1}^{N}\frac{1}{\alpha_r}\ln\left|\frac{\left(z_I^{(r)} - Z_0\right)\left(Z_r - Z_{N+1}\right)}{\left(Z_r - Z_0\right)\left(z_I^{(r)} - Z_{N+1}\right)}\right| \\
& \qquad - \sum_I \frac{1}{\partial^2\rho(z_I)}\left(\frac{1}{z_I - Z_0} - \frac{1}{z_I - Z_{N+1}}\right)\frac{\partial W}{\partial z_I} \\
& \qquad - \frac{3}{2}\sum_I \frac{1}{\partial^2\rho(z_I)}\left(\frac{1}{\left(z_I - Z_0\right)^2} - \frac{1}{\left(z_I - Z_{N+1}\right)^2}\right) \\
& \qquad - \frac{3}{2}\sum_I \frac{\partial^3\rho\left(z_I\right)}{\left(\partial^2\rho\left(z_I\right)\right)^2}\left(\frac{1}{z_I - Z_0} - \frac{1}{z_I - Z_{N+1}}\right)\Bigg]\alpha_0 + \mathcal{O}\left(\alpha_0^2\right). \qquad \text{(D.5)}
\end{aligned}
$$

We can see that $\lim_{\alpha_0\to 0}\Gamma[\ln\left(\partial\rho'\bar{\partial}\bar{\rho}'\right)]$ is divergent and does not coincide with $\Gamma[\ln\left(\partial\rho\bar{\partial}\bar{\rho}\right)]$. This singularity can be avoided by modifying $\Gamma \to \Gamma - \sum_r\left(3\ln|\alpha_r| - 2\right)$, which corresponds to a renormalization of the operator $e^{-ip_r^+X^-}$. Such a renormalization is irrelevant to the

JHEP12(2009)010

calculation of the right hand side of eq. (3.11) and we obtain

$$\partial_{Z_0}\partial_{\alpha_0}\,\Gamma[\ln(\partial\rho'\bar{\partial}\bar{\rho}')]\big|_{\alpha_0=0} = \partial_{Z_0}\left[\sum_{r=1}^{N}\frac{1}{\alpha_r}\ln\left(\frac{z_I^{(r)}-Z_0}{Z_r-Z_0}\right)-\sum_I\frac{1}{\partial^2\rho(z_I)}\frac{1}{z_I-Z_0}\frac{\partial W}{\partial z_I}\right.$$
$$\left.-\frac{3}{2}\sum_I\frac{1}{\partial^2\rho(z_I)}\left(\frac{1}{(z_I-Z_0)^2}+\frac{\partial^3\rho(z_I)}{\partial^2\rho(z_I)}\frac{1}{z_I-Z_0}\right)\right]. \tag{D.6}$$

From this equation, we can compute the right hand side of eq. (3.11) and obtain eq. (3.13).

Let us evaluate the right hand side of eq. (3.16). This can be evaluated by using eq. (D.6) with Z_0, $\rho(z_I)$ and I replaced by z, $\rho'(z'_{I'})$ and I' respectively and with the range of the index r taken to be from 0 to $N+1$. The terms in which we are interested are the $r=0$ contribution of the first term and the $I'=I^{(0)}$ case of the second and the third terms in the square brackets on the right hand side of eq. (D.6) with the replacements mentioned above. In order to evaluate the first term, we use

$$\frac{1}{\alpha_0}\ln\frac{z-z'_{I(0)}}{z-Z_0} = \frac{1}{\partial\rho(Z_0)(z-Z_0)}$$
$$+\left[\frac{-\frac{1}{2}}{(\partial\rho(Z_0))^2}\frac{1}{(z-Z_0)^2}+\left(\frac{\partial^2\rho(Z_0)}{(\partial\rho(Z_0))^3}+\frac{1}{(\partial\rho(Z_0))^2}\frac{1}{Z_0-Z_{N+1}}\right)\frac{1}{z-Z_0}\right]\alpha_0$$
$$+\mathcal{O}\left(\alpha_0^2\right)\ , \tag{D.7}$$

which follows from eq. (D.1). For the computation of the second and the third terms, we use

$$\frac{1}{\partial^2\rho'\left(z'_{I(0)}\right)} = -\frac{\alpha_0}{(\partial\rho(Z_0))^2}-\left[\frac{3\partial^3\rho(Z_0)}{(\partial\rho(Z_0))^4}+\frac{2}{(\partial\rho(Z_0))^3}\frac{1}{Z_0-Z_{N+1}}\right]\alpha_0^2+\mathcal{O}\left(\alpha_0^3\right)\ ,$$
$$\frac{1}{z-z'_{I(0)}} = \frac{1}{z-Z_0}-\frac{1}{\partial\rho(Z_0)}\frac{1}{(z-Z_0)^2}\alpha_0+\mathcal{O}\left(\alpha_0^2\right)\ ,$$
$$\frac{\partial W'}{\partial z'_{I(0)}} = -\frac{\partial^3\rho'(z'_{I(0)})}{\partial^2\rho'(z'_{I(0)})} = -2\partial\rho(Z_0)\frac{1}{\alpha_0}+\frac{2}{Z_0-Z_{N+1}}+\mathcal{O}(\alpha_0)\ . \tag{D.8}$$

Combining these relations, we obtain eq. (3.17).

References

[1] M. Kaku and K. Kikkawa, *The field theory of relativistic strings, pt. 1. Trees*, *Phys. Rev.* **D 10** (1974) 1110 [SPIRES];
M. Kaku and K. Kikkawa, *The field theory of relativistic strings, pt. 2. Loops and Pomerons*, *Phys. Rev.* **D 10** (1974) 1823 [SPIRES].

[2] S. Mandelstam, *Interacting string picture of dual resonance models*, *Nucl. Phys.* **B 64** (1973) 205 [SPIRES].

[3] E. Cremmer and J.-L. Gervais, *Infinite component field theory of interacting relativistic strings and dual theory*, *Nucl. Phys.* **B 90** (1975) 410 [SPIRES].

JHEP12(2009)010

[4] Y. Baba, N. Ishibashi and K. Murakami, *Light-cone gauge superstring field theory and dimensional regularization*, *JHEP* **10** (2009) 035 [arXiv:0906.3577] [SPIRES].

[5] S. Mandelstam, *The interacting string picture and functional integration*, in *Workshop on unified string theories*, M.B. Green and D.J. Gross eds., World Scientific, Singapore (1986), pag. 46.

[6] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory. Vol. 2: loop amplitudes, anomalies and phenomenology*, Cambridge Monographs on Physics, Cambridge University Press, Cambridge U.K. (1987), pag. 596 [SPIRES].

[7] E. D'Hoker and S.B. Giddings, *Unitary of the closed bosonic Polyakov string*, *Nucl. Phys.* **B 291** (1987) 90 [SPIRES].

[8] H. Kunitomo and K. Suehiro, *A new method of constructing gauge string field theory*, *Nucl. Phys.* **B 289** (1987) 157 [SPIRES].

[9] H. Sonoda, *Functional determinants on punctured Riemann surfaces and their application to string theory*, *Nucl. Phys.* **B 294** (1987) 157 [SPIRES].

[10] E. D'Hoker and D.H. Phong, *Functional determinants on Mandelstam diagrams*, *Commun. Math. Phys.* **124** (1989) 629 [SPIRES].

[11] T. Banks, M.R. Douglas, N. Seiberg and S.H. Shenker, *Microscopic and macroscopic loops in nonperturbative two-dimensional gravity*, *Phys. Lett.* **B 238** (1990) 279 [SPIRES].

[12] A. LeClair, M.E. Peskin and C.R. Preitschopf, *String field theory on the conformal plane. 1. Kinematical principles*, *Nucl. Phys.* **B 317** (1989) 411 [SPIRES].

[13] T. Kugo and B. Zwiebach, *Target space duality as a symmetry of string field theory*, *Prog. Theor. Phys.* **87** (1992) 801 [hep-th/9201040] [SPIRES].

JHEP12(2009)010